PROBLEMS OF THE GENERAL THEORY OF PLASTICITY

(VOPROSY OBSHCHEI TEORII PLASTICHNOSTI)

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1. The isotropy postulate [1]. The general mathematical theory of plasticity is being developed primarily for solid bodies, whose material in the undeformed state is isotropic or quasi-isotropic (polycrystalline), obeys Hooke's law in the elastic region and in which the formation of plastic deformations is characterized by a plasticity condition which coincides with sufficient accuracy with the Huber-Mises condition (as, for example, the Tresca and other conditions, which replace the Mises surface by polygones which are close to it, etc.). For the sake of brevity such bodies will be called isotropic in the initial state.

In approaching the problem of stress-strain relations, which determine the basis of the theory, the various plasticity conditions of the indicated type may be called approximate representations of the Mises condition in all those cases in which the ensuing consequences do not differ essentially.

Let us consider a fixed physical point of the body (in the usual sense of continuum mechanics) and a system of orthogonal coordinates (1, 2, 3). The state of stress and strain in the vicinity of the point at the instant of time t and for small deformations is characterized by the stress deviator $\sigma_{ij}(t)$, the strain deviator $\vartheta_{ij}(t)$, the mean stress $\sigma(t)$ and the mean elongation $e(t)$, whereby all these quantities are assumed to be equal to zero in the initial state (for $t = 0$). The state of stress and strain in the vicinity of the point M is always homogeneous, and therefore this point may always be associated with a body T of arbitrary shape and made from the same material, being in the same homogeneous state of stress and strain and the same external conditions (i.e. a specimen, subjected to a test). The small vicinity of the point M is assumed, however, to be sufficiently large, such that the laws for identical processes in M and T (i.e. for example, processes with like given functions $\vartheta_{ij}(t)$, $e(t)$) be identical.

Taking into account the linear dependence of the components э_{ij} ($\text{э}_{ii} = 0$) and of the components σ_{ij} ($\sigma_{ii} = 0$), as well as their physical nonhomogeneity (elongation, shears; normal and shear stresses), the author [1] introduced five-dimensional orthogonal cartesian strain vectors $\mathbf{э} = \text{э}_n \mathbf{e}_n$ (where $\mathbf{e}_n$ is a unit base vector) and the stress vectors $\boldsymbol{\sigma}(\sigma_n)$, as well as the corresponding spaces, with the usual laws of addition and scalar multiplication in each one of them. These will be used below. The components of the strain deviator э_{ij} are expressed in terms of the components of the vector $\mathbf{э}$ by means of the formulas

$$
\begin{aligned}
&\text{э}_{11}\sqrt{\tfrac{3}{2}} = \text{э}_1 \cos\beta + \text{э}_2 \sin\beta, && \text{э}_{12}\sqrt{\tfrac{3}{2}} = \text{э}_3 \cos\tfrac{1}{6}\pi \\
&\text{э}_{22}\sqrt{\tfrac{3}{2}} = -\text{э}_1 \sin\left(\beta + \tfrac{1}{6}\pi\right) + \text{э}_2 \cos\left(\beta + \tfrac{1}{6}\pi\right) && \text{э}_{23}\sqrt{\tfrac{3}{2}} = \text{э}_4 \cos\tfrac{1}{6}\pi \qquad (1.1) \\
&\text{э}_{33}\sqrt{\tfrac{3}{2}} = +\text{э}_1 \sin\left(\beta - \tfrac{1}{6}\pi\right) - \text{э}_2 \cos\left(\beta - \tfrac{1}{6}\pi\right) && \text{э}_{31}\sqrt{\tfrac{3}{2}} = \text{э}_5 \cos\tfrac{1}{6}\pi
\end{aligned}
$$

where β is an arbitrary constant number. The components σ_{ij} are related to σ_n by means of exactly the same formulas, with the same β.

It follows from (1.1) that if some five-dimensional vector $\mathbf{z}$ is related to $\mathbf{э}$ by means of a linear dependence $\mathbf{z} = L(\ )$, then the components of the tensor z_{ij} are related to э_{ij} by means of formulas $z_{ij} = L(\text{э}_{ij})$.

The process of deformation at the point M (or in the body T) is represented in course of time in the space $Э_5$ by means of the strain trajectory $\mathbf{э}(t)$. The scalar geometric characteristics of the strain trajectory (modulus э, are length s, $ds = |d\mathbf{э}|$, four parameters of curvature and torsion κ) are the invariants of the tensor э_{ij}:

$$
|\mathbf{э}| = \sqrt{\text{э}_{ij}^{\,2}}, \qquad ds = \sqrt{d\text{э}_{ij}^{\,2}}, \qquad \varkappa_1 \equiv \left|\frac{d^2\mathbf{э}}{ds^2}\right|, \ldots \qquad (1.2)
$$

Two processes of deformation (referred to the same physical axes (1, 2,3) are called identical (and are physically identical) if their trajectories in $Э_5$ coincide identically and if at the corresponding geometric points of the trajectories the velocities ds/dt are the same, in other words if the times t coincide. If the physical properties of the body do not depend on time explicitly (phenomena of the creep and relaxation type are absent), then the last requirement in the characterization of identical processes is removed.

At each point of the strain trajectory, using, generally speaking, five linearly independent vector derivatives $d\mathbf{э}/ds, \ldots, d^5\mathbf{э}/ds^5$, it is possible to construct a unique orthogonal cartesian set of unit base vectors

$$
\mathbf{p} = \mathbf{p}_1 = \frac{d\mathbf{э}}{ds}, \qquad \mathbf{p}_2 = \frac{1}{\varkappa_1}\frac{d^2\mathbf{э}}{ds^2}, \ldots, \mathbf{p}_5
$$

determined merely by the internal geometry of the strain trajectory. The generalized Frenet formulas

$$\frac{d\mathbf{p}_n}{ds} = \varkappa_{ni}\mathbf{p}_i, \qquad \varkappa_{ni} = \begin{cases} 0 & (i \neq n-1, n+1) \\ -\varkappa_{n-1} & (i = n-1) \\ \varkappa_n & (i = n+1) \end{cases} \tag{1.3}$$

where κ_{ni} is the antisymmetric curvature tensor ($\kappa_0 \equiv \kappa_5 \equiv 0$), permit us to express any derivative $d^m\mathbf{э}/ds^m$ (for $m = 0$ and $m > 5$) in terms of $\mathbf{p}_1, \ldots, \mathbf{p}_5$ and some scalar quantities (the curvature parameters $\kappa_1, \ldots, \kappa_4$ and their derivatives with respect to s). It follows that any operator $L(\mathbf{э})$, linear with respect to the vector $\mathbf{э}$, may be represented by a five-term formula along the arc s or the time t.

$$L(\mathbf{э}) = P_1\mathbf{p}_1 + \ldots + P_5\mathbf{p}_5 \tag{1.4}$$

where P_n are scalars (depending on $s, \kappa, d\kappa/ds, \ldots$).

In as much as under given external conditions (heating and other forms of penetrating action) there corresponds physically to each point of the given strain trajectory its own determined stress vector σ, it can be constructed formally at each point of the strain trajectory, assuming this point to be the initial one with respect to σ, and plotting the components σ_n at a certain scale in the corresponding directions of the base set $\mathbf{e}_n$, i.e. it is possible to write down the vector σ in the form $\sigma = \sigma_1\mathbf{e}_1 + \ldots \sigma_5\mathbf{e}_5$. Thereby, one can determine in an equally formal manner the elementary "work" of the stress vector σ done in the strain trajectory along the path $d\mathbf{э}$, i.e. $\sigma d\mathbf{э}$, which coincides with the physical work of the internal forces σ_{ij} during the interval of time dt in a unit volume of the body

$$\sigma d\mathbf{э} \equiv \sigma_n d\mathbf{э}_n = \sigma_{ij} d\mathbf{э}_{ij} \tag{1.5}$$

The integral $\int \sigma d\mathbf{э}$ along the trajectory gives the total work during the time t. The stress vector σ may also be represented with respect to the base set $\mathbf{p}_n$

$$\sigma = S_n\mathbf{p}_n, \qquad S_m = \sigma_n\mathbf{e}_n \cdot \mathbf{p}_m \tag{1.6}$$

and S_n may be called the natural components of the stress vector. The work will then be given by

$$\sigma d\mathbf{э} = S_1 ds$$

The complete strain trajectory, together with the stress vectors and other physical vectors constructed at all its points, we shall call the pattern of the process of deformation of the body in $Э_5$. The patterns of processes of simultaneous deformation at different points of the body in ∂_5 are different (in a common system of cartesian coordinates (1, 2, 3), and in initially like bodies they coincide identically for like external conditions only in the case of homogeneous deformation of the body.

In addition to a certain arbitrary fixed-strain trajectory one could consider also the totality of all other trajectories in ∂_5 which possess the same or the sign-reversed curvature at corresponding points (κ_n for same s are the same or of opposite sign). All these trajectories may be reduced to a single one by means of linear transformations of rotation and reflection: $\mathfrak{a} = (\alpha_{mn})\mathfrak{a}'$, whereby the quadratic orthogonal normalized matrix of cosines will take on arbitrary values (independent of t), and will have a determinant $|a_{mn}| = +1$ for rotations and $|a_{mn}| = -1$ for reflections.

The transformation of a pattern of a process of deformation in by means of matrix (a_{mn}) (simultaneous transformation of vectors , $\sigma\ldots$) is described as rotation and reflection of the pattern.

Numerous considerations and facts indicate that the physical properties of initially isotropic bodies, more exactly the relationship between the vectors σ and $\mathfrak{a}$, are in accord with the isotropy postulate which states: the pattern of the process of deformation is invariant with respect to the transformations of rotation and reflection, i.e. the representation of the stress vector in the natural base set $\mathbf{p}_n$ of the strain trajectory is invariant with respect to these transformations. This means that the components in (1.6) are transformation invariants, i.e. they depend (possibly in a very complicated manner, for example sourcelike) only on the arc length s, the curvatures κ_n of the trajectory, and (in the presence of properties of the type creep-relaxation) on the velocity ds/dt.

The postulate of isotropy, as applied to plasticity, was verified by tests of Lenskii [2,3] and several others; Lenskii observed rather complex nonanalytical trajectories with many corner points on the trajectory, including unloading and secondary loading.

2. Isomorphism. Instead of the basic space ∂_5 one can take the stress space σ_5, in which the loading process is determined by $\sigma(t)$ and the pattern of a definite loading process may be called the stress trajectory, together with the strain vector $\mathfrak{a}$ and other quantities, constructed at each of its points, for example with respect to the natural set $\mathbf{q}_n$

$$\mathbf{q} = \mathbf{q}_1 = \frac{d\sigma}{d\Sigma}, \qquad \mathbf{q}_2 = \frac{1}{k_1}\frac{d^2\sigma}{d\Sigma^2}, \ldots, \mathbf{q}_5; \; d\Sigma = |d\sigma|, \qquad k_1 = \left|\frac{d^2\sigma}{d\Sigma^2}\right|, \ldots$$

The postulate of isotropy may now be formulated also in the space σ_5; the pattern of the process in σ_5 is invariant with respect to transformations of rotation and reflection, i.e. in the representation

$$\mathbf{э} = E_1\mathbf{q}_1 + \ldots + E_5\mathbf{q}_5 \tag{2.1}$$

the coefficients depend only on the invariants Σ and k.

In general, one can determine two linearly independent five-dimensional vectors

$$\mathbf{l} = L(\mathbf{э}, \boldsymbol{\sigma}), \qquad \mathbf{l}' = L'(\mathbf{э}, \boldsymbol{\sigma}) \tag{2.2}$$

where L, L' are linear operators with respect to э and $\boldsymbol{\sigma}$ (not their invariants), whereby $\mathbf{l}$, $\mathbf{l}'$ are invariant with respect to the transformations of rotation and reflection with the same matrix (a_{ij}). The pattern of the loading process may be, for example, the trajectory of the vector $\mathbf{l}(t)$ in l_5, and the vectors $\mathbf{l}'$ constructed at each of its points, whereby the natural base set $\mathbf{r}_1, \ldots, \mathbf{r}_5$ is obtained from $\mathbf{l}(t)$ just as $\mathbf{p}_1, \ldots, \mathbf{p}_5$ is obtained from $\mathbf{э}(t)$, or $\mathbf{q}_1, \ldots, \mathbf{q}_5$ from $\boldsymbol{\sigma}(t)$, which means that it is determined by the invariants $d\lambda = |d\mathbf{l}|$, $K_1 = |d^2\mathbf{l}/d\lambda^2| \ldots$ The isotropy postulate, in this case, asserts the invariance of the pattern of the process in l_5, i.e. its invariance with respect to transformations with the aid of the matrix (a_{ij}); thus

$$\mathbf{l}' = \Lambda_1\mathbf{r}_1 + \ldots + \Lambda_5\mathbf{r}_5 \tag{2.3}$$

where $\Lambda_1, \ldots \Lambda_5$ depend only on the curvature K and the arc length λ.

There are no principal reasons to make a preferred selection among the representations of the law connecting $\boldsymbol{\sigma}$ and э in the infinite variety of forms (1.6), (2.1) and (2.3) with invariant coefficients S_n, E_n, Λ_n and the parameters s, κ, Σ, k, λ, K; they are all obtained from the isotropy postulate in the corresponding space. Thus, it is natural to formulate the following theorem.

Theorem of isomorphism. For one and the same material whose process of deformation begins and proceeds under like external conditions, the isotropy postulate is equally valid in the spaces of strain $Э_5$, of stress σ_5 and the derived spaces l_5. This means that the relations $\boldsymbol{\sigma} \sim$ э in different forms (1.6) (2.1) and (2.3) are identical, in the sense that they yield one and the same physical law, provided the initial conditions (at the point $t = 0$, $\sigma = 0$, $э = 0$) are the same.

The proof of the isomorphism theorem is based on the fact that in any one space for example in $Э_5$, the isotropy postulate is valid. By means of quintuple differentiation of (1.6) we find the base set $\mathbf{q}_1, \ldots, \mathbf{q}_5$ using Frenet's formula for $\mathbf{p}_n$

$$\mathbf{q}_1 d\Sigma = \left(S_n \frac{dp_n}{ds} + p_n \frac{dS_n}{ds}\right) ds = S_n{}^{(1)}\mathbf{p}_n\, ds, \ldots$$

and the relation between the invariants (s, κ, Σ, k), expressing the $\mathbf{p}_n$ through $\mathbf{q}_n$, is found thereafter as

$$\mathbf{Э} = \int_0^s \mathbf{p}_1\, ds = \int_0^L E_n{}^{(1)}\mathbf{q}_n d\Sigma$$

and therefore also (2.1), since on the right-hand side there is a linear operator with respect to σ, which, as was already indicated above (1.4), may be represented with respect to the base set $\mathbf{q}_n$. Having now two representations (1.6) and (2.1) and constructing from these the vectors $\mathbf{I} = L(Э, \sigma)$ and $\mathbf{I}' = L'(Э, \sigma)$, one can find the base sets $\mathbf{r}_n$ and $\mathbf{r}_n'$, and, eliminating $\mathbf{p}_n$, $\mathbf{q}_n$, to represent $\mathbf{I}'$ with respect to the base set $\mathbf{r}_n$ in accordance to (2.3), i.e. to determine Λ_n; the actual carrying-out to completion of these transformations will meet with difficulties, because the nonlinear relations between the invariants s, κ and Σ, k, λ, K may be implicit and functional. The requirements of reciprocal single-valuedness of the representations (1.6), (2.1) and (2.3) will impose limitations on the dependence of the components S_n, E_n, Λ_n on s and κ, Σ and k and λ and K, respectively, but these limitations are natural.

3. Some consequences of the isotropy postulate. The relations between the stresses σ_{ij} and the strains $Э_{ij}$ for macro-volumes of initially isotropic bodies, which are sometimes called the mechanical equations of state, are divided into vector and scalar ones [1]. The isotropy postulate determines the vectorial properties and reduces the problem of determining the relations between σ and to the determination of scalar properties only. For all cases of simple loading, independent ly of the rheological properties of the body, the isotropy postulate yields a vector law, which can always be reduced to the most simple form

$$\boldsymbol{\sigma} = \frac{\sigma}{Э}\mathbf{Э} \qquad \left(\sigma_{ij} = \frac{\sigma}{Э}Э_{ij}\right) \tag{3.1}$$

which means that the scalar properties can be determined completely by the most simple tests (the behavior of the specimen under simple extension or torsion, etc.) which determine a single unknown function σ in terms of $Э$, $dЭ/dt$. Formula (3.1) establishes a general physical law for initially isotropic materials subjected to simple (proportional) loading, [4].

For metals we obtain in cases of trajectories with small curvature

[1], when κ_{ni} from (1.3) satisfies the condition $|\kappa_{ni}| < 1/fe_s$ (e_s is the limiting elastic elongation, $f \approx 4 \sim 10$) from the postulate of isotropy, as a consequence of the after effect [1,2], $\sigma = S_1 \mathbf{p}_1$ or

$$\sigma = \sigma \frac{d\mathfrak{s}}{ds} \quad \left(d\mathfrak{s}_{ij} = \frac{ds}{\sigma} \sigma_{ij}\right) \tag{3,2}$$

whereby this law contains (3.1), because for simple loading $d\mathfrak{s}/ds = \mathfrak{s}/\mathfrak{s}$. From what was said above, the difference between (3.1), (3.2) and both the "deformational theory" and the most simple theory of plastic flow becomes clear: relations (3.1) and (3.2) represent a general theory of plasticity for definite classes of deformational trajectories (simple and almost simple loading and trajectories of small curvature).

For arbitrary analytical trajectories of deformation the isotropy postulate and the isomorphism theorem yield the relation (1.6) between σ and $\mathfrak{s}$ which may be written down in the form of a five-term formula

$$\sigma = \sum_{n=N}^{N+4} F_{Nn} \frac{d^n \mathfrak{s}}{ds^n} \tag{3.3}$$

Here N is an arbitrary integer, F_{Nn} are universal functions (operators, functionals) which depend on s and the curvatures at the point s or at the point $s = s_0 =$ const (s_0 is an arbitrary fixed point).

If the trajectory is not an analytical curve, but consists of pieces of analytical curves interconnected by nonanalytical points (for example, corners), then (3.3) is conserved on each piece; it is thereby assumed of course that the vectors $d^n \mathfrak{s}/ds^n$ entering into (3.3) are linearly independent.

In the general case of nonanalytical trajectories, the derivatives in (3.3) should be replaced by difference relations; if at the point s, using the preceding portion of the trajectory (in the region of plastic deformation) one can construct only $m > 5$ linearly independent vectors, only an m-term formula will be used to express σ in terms of $\mathfrak{s}$.

In principle, the relation between the macro-stress σ and strain $\mathfrak{s}$ may be found theoretically; a physical investigation should yield the mechanics of plastic deformation of a crystal (or a molecule) of the material, statistically mechanics should lead to equations of state and hence give the sought relation $\sigma \sim \mathfrak{s}$. In as much as the isotropy postu late is in accord with macroscopic tests, the deductions from the isotropy postulate and the molecular theory will coincide (of course, with a certain accuracy), whereby the deductions from the latter theory will be broader, because they will give not only the vectorial but also the scalar properties, i.e. the form of the functions F_{Nn} in (3.3); the noncoincidence of the deductions would indicate that in the unavoidable multitude of assumptions, of both physical and mathematical character in the theoretical deduction of the equation of state, inaccuracies are contained. From our point of view, the following semi-inverse problem is of interest: the theory of probability of weakly interacting random processes is known, the probable character of grain distribution with respect to size and mutual crystallographic orientation is known, certain properties of plasticity mechanism in the grain are known (allowable shears, etc.): what is the class of supplementary physical information, possessing the property that the macrorelations $\sigma \sim \text{э}$ satisfy the isotropy postulate? The solution of this problem would clarify new properties of plastic microdeformations and the methods of their statistical treatment.

4. Elasto-plastic properties for loading along a broken line (with one corner point). Let in strain space ∂_5 the strain trajectory represent simple loading along an arbitrary unit vector $\mathbf{p}$ to the point $s = |\text{э}| = \text{э}$, in which at the instant t_0 there occurs a discontinuity of the trajectory and the further process proceeds along an arbitrary vector $\mathbf{p}_1$, whereby at the instant $t > t_0$ the deformation is characterized by the vector э_1 (Fig. 1)

$$\delta s = s_1 - s, \qquad \delta \text{э} = \text{э}_1 - \text{э}, \qquad \mathbf{p} = \frac{\text{э}}{\text{э}}, \qquad \mathbf{p}_1 = \frac{\delta \text{э}}{\delta s} \tag{4.1}$$

Such processes of loading have practical significance and are realized in bodies under simple loading up to the moment of stability loss and the subsequent deformation in the postcritical range. Thereby, the loading may also change abruptly at the instant of loss of stability and thereafter vary smoothly with respect to time. Also other practical questions could be mentioned which may be reduced to the case under consideration.

The rotation of the strain trajectory at the corner point $s = |\text{э}|$ shall be called the quantity

$$\tau = \cos\theta = \frac{\text{э}\delta\text{э}}{\text{э}\delta s} = \frac{\sigma\delta\text{э}}{\sigma\delta s} \tag{4.2}$$

It is obvious that for an arbitrary point on the second portion of the strain trajectory, on the basis of the preceding portion of the trajectory, it is impossible to construct any vectors (by means of linear operations on э and э_1) which would be linearly independent with respect to $\mathbf{p}$ and $\mathbf{p}_1$. From the isotropy postulate it follows that the single most general possible stress expression σ_1 at the point э_1 in terms of strains will be the two-term law

$$\delta\sigma = (S\mathbf{p} + S_1\mathbf{p}_1)\,\delta s \qquad (\delta\sigma = \sigma_1 - \sigma) \tag{4.3}$$

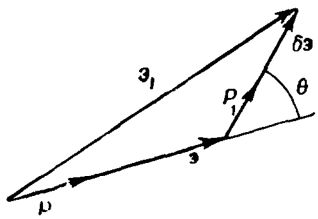

FIG. 1.

and S, S_1 will be functions of the trajectory invariants. The trajectory invariants are only the quantities s, $\delta s = s_1 - s$ and $\tau = \cos\theta$, whereby $\sigma = (\sigma/э)э$ and $\sigma = \Phi(э)$ is a function, known for simple loading. Therefore, the coefficients S and S_1 are functions only of the lengths of the portions s and δs and the rotation τ.

$$S = S(s, \tau, \delta s), \qquad S_1 \equiv N = N(s, \tau, \delta s) \tag{4.4}$$

Let us introduce a normal unit vector $\mathbf{n}$ as a vector, lying in the plane of the vectors $\mathbf{p}$ and $\mathbf{p}_1$

$$\mathbf{n} = -\mathbf{p}\operatorname{ctg}\theta + \mathbf{p}_1\frac{1}{\sin\theta}, \qquad \mathbf{p}_1 = \mathbf{p}\cos\theta + \mathbf{n}\sin\theta \tag{4.5}$$

Then (4.3) may be represented in the form

$$P = N + \frac{S}{\cos\theta} = N + \frac{S}{\tau} \tag{4.6}$$

Therefrom, designating for brevity the projections $\delta э$ and $\delta\sigma$ on $\mathbf{p}$ and $\mathbf{n}$ by

$$\delta\boldsymbol{\sigma} = (P\cos\theta\,\mathbf{p} + N\sin\theta\,\mathbf{n})\,\delta s$$

we find the expressions for P and N

$$P = \frac{\delta\sigma_p}{\delta s_p}, \qquad N = \frac{\delta\sigma_n}{\delta s_n} \tag{4.8}$$

These expressions (as we shall see), permit us to find the functions P and N of the arguments s, δs, τ from simple tests. Now, taking into account the relations

$$\mathbf{p}_1\delta s = \delta\mathbf{э}, \qquad \mathbf{p}\,\delta s\cos\theta = \mathbf{p}(\mathbf{p}\delta\mathbf{э}) = \frac{\boldsymbol{\sigma}}{\sigma^2}(\boldsymbol{\sigma}\delta\mathbf{э})$$

we write down the stress-strain relation (4.3) in the form

$$\delta\boldsymbol{\sigma} = N\delta\mathbf{э} - (N - P)\frac{\boldsymbol{\sigma}}{\sigma^2}(\boldsymbol{\sigma}\cdot\delta\mathbf{э}) = N\delta\mathbf{э} - (N - P)\frac{\boldsymbol{\sigma}}{\sigma}\tau\delta s \tag{4.9}$$

whereby N, P are considered to be known functions of s, δs and τ and therefore (4.9) represents the most general law of the relation $\sigma \sim \mathfrak{I}$ for a broken line. The following theorem becomes obvious.

Theorem. The functions N and P are fully determined from tests of combined extension and torsion of thin-walled tubes and coincide identically with functions determined by experiments of the "fan" type.

The tests of Lenskii on the automatic testing machine of the Institute of Mechanics of the Academy of Sciences of the USSR realize a strain program: torsion of the tube up to different degrees of deformation (direction $\mathbf{e}_3$, component of the strain vector $\mathfrak{I}_3$) and at each degree of deformation further continued torsion (including strain reversal) and extension along a rectilinear ray at an angle θ with respect to the axis $\mathbf{e}_3$; for extension, the direction $\mathbf{e}_1$, the component of the strain vector $\mathfrak{I}_1$, whereby at each corner point the fan of straight lines for different angles of inclination θ is constructed; for each broken line a new specimen is used. Graphs are constructed and the properties of the functions N and P are investigated.

The theorem follows from the fact that any space trajectory may be transferred from the plane $(\mathbf{p}, \mathbf{p}_1)$ to the plane $(\mathbf{e}_1, \mathbf{e}_3)$ by means of a rotation transformation and is therefore a consequence of the isotropy postulate, The functions

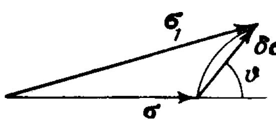

FIG. 2.

$$P_1 = \frac{\delta\sigma_1}{\delta\mathfrak{I}_1}, \qquad N_1 = \frac{\delta\sigma_3}{\delta\mathfrak{I}_3}$$

are determined by the arguments $\mathfrak{I}_1$, θ and δs

$$\delta s = \sqrt{\delta\mathfrak{I}_1^2 + \delta\mathfrak{I}_3^2}, \quad \delta\mathfrak{I}_1 = \delta s\cos\theta, \quad \delta\mathfrak{I}_3 = \delta s\sin\theta$$

on the basis of measurement of the quantities $\delta\sigma_1 = \sigma_1' - \sigma_1$, $\delta\sigma_3 = \sigma_3'$ $(\sigma_3 = 0)$. The passage to P and N from (4.9) is obtained by replacing $\mathfrak{I}$ by s in $P_1 N_1$ and representing θ in the form (4.2).

The law (4.9) furnishes the expressions for the stresses in forms of deformations. Let us find the inverse relations, i.e. the representation of the law in stress space (isomorphism theorem). Squaring the left- and right-hand sides of (4.9) and designating the arc differential of the corresponding trajectory in σ_5-space by $\delta\Sigma$, we note that, generally speaking, $|\delta\sigma|$ will not be equal to $\delta\Sigma$, because the rectilinear portion after the break of the trajectory in $\mathfrak{I}_5$ will not be rectilinear in σ_5 and therefore $\delta\Sigma > |\delta\sigma|$ (Fig. 2); we obtain the following expression

$$|\delta\sigma| = \delta s\sqrt{N^2 + (P^2 - N^2)\tau^2}, \qquad |\delta\sigma| \leqslant \delta\Sigma \tag{4.10}$$

Multiplying now (4.9) by σ, we obtain $\sigma\delta\sigma = P\sigma\delta\mathfrak{I}$. Let us designate the angle between σ and $\delta\sigma$ by ϑ and its cosine by t

$$t = \cos\vartheta = \frac{\sigma\delta\sigma}{\sigma|\delta\sigma|} \tag{4.11}$$

We obtain then a second relation connecting $|\delta\sigma|$, ϑ with δs, θ

$$t = \tau\frac{P\delta s}{|\delta\sigma|} = \frac{\tau P}{\sqrt{N^2+(P^2-N^2)\tau^2}} \tag{4.12}$$

Relations (4.10) and (4.12) express $|\delta\sigma| = |\sigma_1 - \sigma|$ and $t(\vartheta)$ in terms of P, N, τ and δs, i.e. in terms of $\vartheta = \Phi^{-1}(\sigma)$, δs and $\tau(\theta)$, and therefore for strain-hardening materials they make it possible to express inversely P, N, τ and δs in terms of σ, $|\delta\sigma|$ and $t = \cos\vartheta$.

Assuming that such a transformation was performed, and solving (4.9) with respect to the strain ∂, we obtain finally the transformed law

$$\delta\partial = \frac{1}{N}\delta\sigma + \frac{N-P}{NP}\frac{\sigma}{\sigma}t|\delta\sigma| = \frac{\delta\sigma}{N} + \frac{N-P}{NP}\frac{\sigma}{\sigma^2}(\sigma\delta\sigma) \tag{4.13}$$

For the fixed broken line considered in ∂_5, the angle θ and τ remain constant in increasing δs; since from (4.12) for τ = const, P and N change with an increase of δs, the angle ϑ and the inclination $t = \cos\vartheta$ will change during the passage along the portion after the corner point in σ_5, i.e. the second portion of the trajectory will be curvilinear.

Expanding the functions $P(s, \tau, \delta s)$, $N(s, \tau, \delta s)$ into a series of

$$P = P_0(s,\tau) + \delta s\frac{\partial P}{\partial(\delta s)} + \ldots, \qquad N = N_0(s,\tau) + \delta s\frac{\partial N}{\partial(\delta s)} + \ldots$$

we obtain the following for small δs

$$|\delta\sigma| = \delta\Sigma, \qquad t_0 = \frac{\sigma\delta\sigma}{\sigma\delta\Sigma} = \frac{\tau P_0}{\sqrt{N_0^2+(P_0^2-N_0^2)\tau^2}} \tag{4.14}$$

$$\delta\Sigma = \delta s\sqrt{N_0^2+(P_0^2-N_0^2)\tau^2}$$

whereby, since from the "fan" P_0, N_0 are determined functions of τ, this means that the first of the formulas (4.14) gives an expression of τ in terms of t, i.e. P_0, N_0 become known functions of t. To the corner point in ∂_5 with rotation τ there corresponds a corner point in σ_5 with rotation t_0. The expressions for the stresses in terms of the strains and the strains in terms of the stresses remain in the form (4.9), (4.13) with a replacement of P, N, t and $|d\sigma|$ by P_0, N_0, t_0 and $\delta\Sigma$.

5. Deformational anisotropy and the expression of elastic strains through stresses. The anisotropy forming during the process of plastic deformations, as well as other mechanical properties, is in accord with the isotropy postulate and its consequences. For simple loading, followed by unloading in any possible rectilinear direction, all formulas of Section 4 remain in force and give the general properties of the elastic deformational anisotropy, since the functions P, N are known from tests for an arbitrary $r = \cos\theta$. As is known, experimental evidence shows that the process of unloading after preceding plastic deformations is not strictly linear, which explains our remark regarding the unloading along rectilinear directions.

However, with a certain satisfactory degree of accuracy, the process of unloading along an arbitrary trajectory is reversible and linear with respect to the stress-strain relation. Under this assumption the elastic properties (deformational anisotropy) after elastoplastic simple loading are completely determined theoretically. Let us attach an index k, at the end point K of the process of simple loading, to all quantities pertaining to this point and let us consider infinitely small increments $d\text{э}$, $d\sigma$, corresponding to an arbitrary unloading path.

These quantities are interrelated by the law (4.9), (4.13) which now takes on the form

$$d\boldsymbol{\sigma} = N d\boldsymbol{\text{э}} - (N - P)\frac{\boldsymbol{\sigma}^k}{\sigma_k^2}(\boldsymbol{\sigma}^k d\boldsymbol{\text{э}})$$

or

$$d\boldsymbol{\text{э}} = \frac{1}{N} d\boldsymbol{\sigma} + \frac{N-P}{NP}\frac{\boldsymbol{\sigma}^k}{\sigma_k^2}(\boldsymbol{\sigma}^k d\boldsymbol{\sigma}) \tag{5.1}$$

whereby the index k is used as a superscript with vectors and as a subscript with scalars. Under our assumptions these relations should be linear with respect to σ and э not only in the vectorial but also in the general sense and therefore N and P should have constant (not dependent on r or σ, t) values, i.e. they should be determinable only by the point K

$$P = P_k, \qquad N = N_k \tag{5.2}$$

In certain versions of the theory of plasticity one considers instead of the general [total] deformation э, the plastic deformation э^p, where э^p is understood to be the difference between the total deformation and the elastic deformation э^e, which is calculated from the stress σ in accordance with Hooke's law for initial (isotropic) state of the body ($\text{э}^e = \sigma/G$); but there is no justification for doing so, because during the process of deformation the body becomes anisotropic and the elastic component of the total deformation is an unknown linear function of stress, and therefore the plastic strain э^p cannot be in principle determined theoretically in terms of э and σ if the path of the preceding loading is not given and if the elastic properties after unloading are not investigated. From the isotropy postulate follows naturally the representation of the increment of plastic deformation in terms of the stress vector, for example in the form (for analytical trajectories)

$$d\mathfrak{э}^p = Q_n \frac{d^n\sigma}{d\Sigma} d\Sigma \quad (n = 0,1,2,3,4) \tag{5.3}$$

where the Q_n will depend in a more or less complicated fashion on the internal geometry of the stress trajectory σ in the space σ_5; thereby $\mathfrak{э}^p$ in an experiment is understood to be the measurable quantity of residual deformation (for unloading, when $\sigma = 0$) and therefore the Q_n may be determined in principle. However, relation (5.3) and other similar ones, in which the increment of the tensor of plastic deformations $d\,\mathfrak{э}_{ij}{}^p$ is expressible in an arbitrary manner in terms of the stress tensor σ_{ij}, do not represent theories of plasticity, i.e. do not permit the formulation mathematically of the problem of deformations of a body in a nonhomogeneous state of stress as long as the expression for elastic strain

$$\mathfrak{э}^e = \mathfrak{э} - \mathfrak{э}^p \tag{5.4}$$

is not given in terms of stress σ. Only this relation, together with (5.3), will establish, in the final analysis, the relation between the components of the stress tensor and the displacement components in the body, i.e. complete the system of equations of equilibrium or of motion.

For elastic deformations also the postulate of isotropy enables us to write down the expression which for analytical trajectories can always lead to one of the forms:

$$\mathfrak{э}^{(e)} = R^k \sigma + R_{n,m}{}^k \left(\sigma \frac{d^n\sigma^k}{d\Sigma_k^n}\right) \frac{d^m\sigma^k}{d\,\Sigma_k^m} \qquad n,m = 0,1,1,3)$$

$$\sigma = S^k \mathfrak{э} + S_{n,m}{}^k \left(\mathfrak{э} \frac{d^n\mathfrak{э}^k}{ds_k{}^n}\right) \frac{d^m\mathfrak{э}^k}{ds_k^m} \qquad (n,m = 0,1,2,3) \tag{5.5}$$

Here the vectors $d^n\sigma^k/d\Sigma_k{}^n$ are relative to a point k of the loading trajectory, the parenthesis contains the scalar product of these vectors and the running vectors σ, $\mathfrak{э}$, while R_{nm}, S_{nm} are parameters of elastic anisotropy, which depend on the curvature of the loading trajectory to the point K, as is indicated by a superscript. It follows from this, among others, that the deformational anisotropy is determined in the most general case by eleven ($R_{mn} = R_{nm}$) elastic "constants", which determine the shear properties of the material. The plastic strain vector $\mathfrak{э}^p$, in accordance with (5.9) and (5.5), is now completely determined, if the "constants" R_{mn} are known:

$$\mathfrak{э}^p = \mathfrak{э} - \mathfrak{э}^e$$

In the investigated case of deformational anisotropy for simple loading the integration of Equation (5.1) under condition

$$\sigma = \sigma^k, \ \mathfrak{э} = \mathfrak{э}^k = \frac{\mathfrak{э}_k}{\sigma_k}\sigma^k = \frac{\Phi^{-1}(\sigma_k)}{\sigma_k}\sigma^k \tag{5.6}$$

gives

$$ə = ə^k - \frac{\sigma}{N_k} + \frac{N_k - P_k}{N_k P_k} \frac{\sigma^k}{\sigma_k^2} (\sigma^k \sigma) - \frac{\sigma^k}{P} \qquad (5.7)$$

Since for $\sigma = 0$ there should be $ə = ə^p$, we find from (5.7) the plastic strain

$$ə^p = ə^k - \frac{\sigma^k}{P_k} = \frac{P_k ə_k - \sigma_k}{P_k} \frac{\sigma^k}{\sigma_k} \qquad (5.8)$$

and subtracting it from (5.7) we obtain the expression of elastic strain, i.e. the law of deformational anisotropy

$$ə^e = \frac{1}{N} \sigma + \frac{N - P_k}{N_k P_k} \frac{\sigma^k}{\sigma_k^2} (\sigma \ \sigma^k) \qquad (5.9)$$

Hence, the tensor of elastic constants of deformational anisotropy depends in a fully determined manner on maximum stress for simple loading σ and on two constants completely determinable by σ_k from simple tests with a thin-walled tube subjected to extension and torsion in the unloading stage. It is seen from (5.9) that P_k is the modulus of elasticity in the direction σ^k, and N_k is the modulus in the direction perpendicular to σ^k.

Solving (5.9) with respect to σ, we obtain

$$\sigma = N_k ə^e - (N_k - P_k) \frac{ə^k}{ə_k^2} (ə^k ə^e) \qquad (5.10)$$

Passing to a coordinate representation, we write (5.9) and (5.10) in the form

$$ə_{ij}^e = \sum \alpha_{ijmn} \sigma_{mn}, \qquad \sigma_{ij} = \sum a_{ijmn} ə_{mn}^e \qquad (5.11)$$

The tensors of the elastic "constants" have thereby the following expressions

$$\alpha_{ijmn} = \alpha_{mnij} = \frac{\delta_{ijmn}}{N_k} + \frac{N_k - P_k}{N_k P_k} \frac{\sigma_{ij}^k \sigma_{mn}^k}{\sigma_k^2} \qquad (5.12)$$

$$a_{ijmn} = a_{mnij} = N_k \delta_{ijmn} - (N_k - P_k) \frac{ə_{ij}^k ə_{mn}^k}{ə_k^2}$$

where $\delta_{ijmn} = 1$ only for simultaneous equality $i = m$, $j = n$ and in all other cases is equal to zero. The invariants are

$$\sigma_k^2 = (\sigma_{ij}^k)^2, \qquad ə_k^2 = (ə_{ij}^k)^2$$

All eleven parameters of elastic anisotropy being formed after an arbitrary loading path up to the point K may be found from the same tests, in which this loading path is realized. To this end it is sufficient, to conduct unloading tests along arbitrary, but different trajectories, measuring each time σ (or $\delta\sigma$) and $ə^e$ (or $\delta ə^e$) and solving the system (5.5) with respect to the unknown R_{mn}.

6. The loading surface, secondary plastic deformations and some particular cases. For each strain trajectory $э(s)$ (in $\mathcal{Э}_5$) and stress trajectory $\sigma(\Sigma)$ (in σ_5), limiting surfaces may be constructed which possess the property that for secondary loading, after unloading from the point K and the appearance of the tip of the vector $э$ (in $\mathcal{Э}_5$) on the limiting strain surface and, which is the same, the appearance of the vector σ (in σ_5) on the limiting stress surface (loading surface), secondary plastic deformations begin to form, i.e. the relation (5.5) is violated. The case of non-strain-hardening materials, when the limiting surface is a sphere $\sigma \equiv \sigma_k = \sigma_s$ (where σ_s is the constant-yield point) and when the space σ_5 degenerates into a four-dimensional space, is not considered here, because in this case a single-valued relation between $э$ and σ does not, in principle, exist.

The most general form of the limiting surface in $\mathcal{Э}_5$ will be

$$\chi \equiv э - \eta(\pi_1, \pi_2, \pi_3, \pi_4) = 0 \tag{6.1}$$

where the invariants π_n are, within factors, the cosines of the angles between the vector and four arbitrary linearly-independent vectors, constructed at the point K; in the case of an analytical trajectory (to the point K) the invariants π_n may be taken, for example, in the form

$$\pi_{n+1} = \frac{э}{э} \cdot \frac{d^n э^k}{ds_k^n} \quad (n = 0, 1, 2, 3), \qquad \text{or} \qquad \pi_n' = p_n^k \cdot \frac{э}{э} \quad (n = 1, 2, 3, 4) \tag{6.2}$$

where p_n^k are four unit vectors of the Frenet pentagon at the point K of the strain trajectory.

In the space σ_5 the equation of the limiting surface for the point K will be of the form

$$\psi = \sigma - f(\rho_1, \rho_2, \rho_3, \rho_4) = 0 \tag{6.3}$$

where ρ_n are four invariants analogous to π_n. For an analytical trajectory (to the point K) the invariants ρ_n may be taken in the form

$$\rho_{n+1} = \frac{\sigma}{\sigma\sigma_{kn}} \cdot \frac{d^n \sigma^k}{d\,\Sigma^n} \quad (n = 0, 1, 2, 3) \quad \left(\sigma_{kn} = \left|\frac{d^n \sigma}{d\,\Sigma_k^n}\right|\right) \tag{6.4}$$

such that the ρ_n are the cosines of the angles between σ and $d^{n-1}\sigma^k / d\Sigma_k^{n-k}$

$$\rho_1 = \rho = \frac{\sigma \cdot \sigma^k}{\sigma\sigma_k}, \qquad \rho_2 = \frac{1}{\sigma\left|\frac{d\sigma^k}{d\,\Sigma_k}\right|}\, \sigma \cdot \frac{d\sigma^k}{d\,\Sigma_k}, \ldots$$

The possibility of representing the surfaces in the form (6.1), (6.3) is a consequence of the isotropy postulate.

It should be taken into account thereby, that the very form of the functions η and f depends on the trajectories $\mathbf{э}(s)$ and $\boldsymbol{\sigma}(\Sigma)$, i.e. Equations (6.1) and (6.3) are functional. Therefore, the passage from the surface ψ to the surface $\psi + \delta\psi$ cannot be obtained by means of formal differentiation of (6.3) with respect to σ and ρ_n, but depends also on the point on the surface (6.3) at which $\delta\psi$ is considered and also on the vector $\delta\boldsymbol{\sigma}$ at this point. And only in the case when the trajectory to the point K is completely determined and the point K is fixed, do the functions η and f become usual functions of the arguments indicated in (6.1), (6.3). Let us explain in greater detail these assertions, to which end we find the normal, for example to the surface ψ, at some of its points $\boldsymbol{\sigma}$:

$$m\mathbf{n} = \operatorname{grad}\psi = \operatorname{grad}\sigma - \frac{\partial f}{\partial\rho_i}\operatorname{grad}\rho_i \qquad (i=1,2,3,4)$$

$$\operatorname{grad}\sigma = \frac{\boldsymbol{\sigma}}{\sigma}, \quad \operatorname{grad}\rho_i = -\frac{\boldsymbol{\sigma}}{\sigma^2}\rho_i + \frac{1}{\sigma\sigma_{k,\,i-1}}\frac{d^{i-1}\bar{\boldsymbol{\sigma}}^k}{d\,\Sigma_k^{i-1}} \tag{6.5}$$

whereby m is the modulus of the right-hand side of (6.5) and therefore $\mathbf{n}$ is the unit vector, normal to the surface $\psi = 0$.

Let us consider an arbitrary small increment $\delta\boldsymbol{\sigma}$ at this same point $\boldsymbol{\sigma}$, having the normal component

$$\delta\sigma_n = \mathbf{n}\cdot\delta\sigma \tag{6.6}$$

On the basis of the definition of the surface $\psi = 0$ the plastic deformations will not increase, i.e. $\delta\boldsymbol{\sigma}$ will be determined from (5.5) if

$$\delta\boldsymbol{\sigma}_n = n\cdot\delta\boldsymbol{\sigma} < 0 \tag{6.7}$$

and, conversely, will increase, if

$$\delta\sigma_n = n\cdot\delta\boldsymbol{\sigma} > 0 \tag{6.8}$$

If the transition through the limiting surface of strain-hardening materials is continuous, then on the surface $\psi = 0$ the condition

$$\mathbf{n}\delta\boldsymbol{\sigma} = 0 \tag{6.9}$$

must be satisfied.

Equation (6.9) represents an identity, which is satisfied by the function f.

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